**Features of Functions**

For this unit you are expected to master the following concepts:

[1.1] know and use the definition of a function; recognize and use function notation;

[1.2] name the domain and range of a function;

[1.3] read and interpret the graph of a function;

[1.4] name and explain increasing and decreasing behaviors of functions;

[1.5] name and explain the intercepts of a function;

[1.6] name and explain the end behaviors of a function;

[1.7] name any extrema (minima or maxima) for a function;

[1.8] given a value for x, find f(x); given a value for f(x), find x.

Here are sample questions for this unit. This is not a full, comprehensive set of problems.

[1] Name the domain and range for f(x), which f(x)

is graphed to the right:

[2] Is f(20) greater than, equal to, or less than f(10)? Explain your response.

[3] Name any minima or maxima for f(x).

[4] Examine the graph of g(x): identify where g(x) is g(x)

increasing and where it is decreasing.

[5] Name the domain and range of g(x).

[6] Name the x- and y-intercepts of g(x).

[7] For g(x), suppose x represents time in minutes

and g(x) represents the temperature of a storage unit.

Name the time that corresponds with a temperature of

0.

[8] Name any minima or maxima for g(x).

[9] Given that h(x) = 2x – 5, find h(-1).

[10] Given that j(x) = $\sqrt{4+3x}$ , find x when j(x) = 1.

Some people learn these skills because they are told to learn them; some people wonder why they have to know them. Go ahead and wonder—but if you want to succeed, figure it out.

 **Lesson 1.1 Functions and Function Notation**

🡪***What is a function***? A function is a special relationship between input values and related output values. For example, the outdoor temperature in your backyard is a function of time:

Let t = time in hours past midnight, and let f(t) = the temperature in degrees Fahrenheit at t.

When t time = 7 | 8 | 9 | 10 | What makes this a function is that for each time t,

then f(t) temp. = 68 | 72 | 77 | 81 | there is only one temperature f(t).

\*\***Here’s the important point: For f(t) to be a function, then for each t value, there is ONLY ONE f(t) value; for any function where y = f(x), for EACH input x, there is ONLY ONE output y.**

f(t) is a function and its graph shows it is a function: no two points of the graph have the same x value. The graph of f(t) passes the **vertical line test**: wherever you draw a vertical line, the line will intersect f(t) at only one point.

If you wonder what f(t) means, it means “the function of t”: it does not mean multiply f by t. When you read f(x) = x2 – 5x, you should say, “The function of x equals x squared minus five x.”

The relation R that is graphed below f(t) is NOT a function. Notice the graph of R does not pass the vertical line test: if you draw a random vertical line that intersects the graph, the line will touch more than one point.

**SAMPLE QUESTIONS**

**Now it’s your turn: you should see if you can answer these sample questions.**

Determine which of the following relations are functions. In each case, explain your thinking.

[1] [2] (-2, 3), (-2, 2), (-1, 1), (0, 0), (1, 1)



 [3] [4]

 **Lesson 1.2 Domain and Range of a Function**

The **domain** of a function is the set of all input values (in many cases, the set of all x values) that you can plug into a function, in order to find a function value.

The **range** of a function is the set of all output values (in many cases, the set of all y values) you can get for a function.

You will be asked to name the domain and range for a given function.

EXAMPLE 1.2 A Name the domain and range for f(x) = $\sqrt{x-3}+5$.

*Solution*: To name the domain for f(x), you need to realize that you cannot take the square root of any number below zero. Therefore you need to set the radicand x – 3 > 0 and solve for x: If x – 3 > 0, then x > -3. Your domain is such that x must be -3 or greater. This is your domain. In set notation, we say {x| x > -3}; in interval notation we say [-3, $\infty $). The square bracket tells you -3 is included, while the parenthesis tells you infinity is not included.

The range of f(x) can be identified by realizing that, since the lowest value possible for $\sqrt{x-3}$ is zero, the lowest value for $\sqrt{x-3}+5$ is 5. f(x) can go higher than 5: it just cannot go below 5. Therefore, for the range we say this: y > 5. In set notation, we say {y| y > 5}; in interval notation we say [5, $\infty $).

EXAMPLE 1.2 B Name the domain and range for g(t), g(t)

which is graphed at the right.

*Solution*: If you look closely, the graph goes left and right without end: we can’t graph the entire function; it continues beyond the page. Therefore the domain is ($-\infty ,\infty $), the set of all real numbers.

The graph is limited in how high it goes: the highest g(t) value is 9. Therefore the range is [9, $\infty $), the set of all numbers 9 and above.

**SAMPLE QUESTIONS**

**Try these questions.** For each function, name the domain and range.

[5] y = x2 – 6. [6] [7] f(x) = $\sqrt{2x+1}$



 **Lesson 1.3 Reading and Interpreting Graphs of Functions**

To the right is the graph of a function that

relates the time of day t to the temperature h(t), in a controlled environment.

Let t represent time in hours, t = 0 represent the time at midnight on March 4, and let h(t) represent the temperature at time t.

You can use the graph to tell you several things:

\*At time t = 0 (midnight, March 4), the temperature h(0) = 56.26o. -10 0 10 20

\*h(4.5) = 47. We say, “h of 4.5 equals 47.” In real English, at 4:30 a.m., the temperature is 47o.

\*If you need to state the temperature at t = 10 (10 hours after midnight), you can estimate this. The x-axis is scaled with five-hour increments, so at t = 10, h(10) appears to be about 60, so at 10:00 a.m., the temperature is about 60o.

\*You can see that the graph shows a minimum temperature of 47o and a maximum temperature of 77o.

\*The function is periodic: it repeatedly goes up and down. This is a trigonometric function.

\*If you look at the change in time t from high point to the next high point, you will see that $∆t$—the change in t—is 24 hours. We can use this change to say that every 24 hours, the temperature returns to what it was in this controlled environment. 🡪This means that when t = 24, the temperature will be 56.26o, the same as at t = 0.

**SAMPLE QUESTIONS**

Use the graph of y = g(x)at the right to answer the following questions.

[8] Find g(-1). [9] Find g(5).

[10] Find x when g(x) = 3.

[11] Find the change in y (called $∆y,$ “delta y”) from x = 0 to x = 2.

[12] Is g(-10) greater than, equal to, or less than g(-5)? Explain your thinking.

 **Lesson 1.4 Increasing and Decreasing Behavior of a Function**

Look at the graph of the function f(x) to the right. Assume that the domain of f(x) is $(-\infty ,\infty )$, the set of all real numbers. Also assume that all trends for f(x) are shown: f(x) continues to climb as x becomes more negative beyond -5, and f(x) continues to fall as x becomes more positive beyond 5.

Read the graph as a book from left to right; in fact, imagine you start at the left and ride the graph as though it is a roller coaster. The roller coaster **decreases** from the far left until x = -4. Then the roller coaster **increases** from x = - 4 until x = -1. The roller coaster decreases again until x = 0; then the roller coaster increases from x = 0 to x = 2. Finally, the roller coaster decreases from x = 2 onward, as x approaches infinity.

These **intervals of increase and decrease** are important in understanding a function. Here, f(x) increases in the intervals $(-4, -1)$ and (0, 2); the function decreases in the intervals $(-\infty ,-4)$, (-1, 0) and $(2,\infty )$.

**You should notice that you are to name the intervals on the *x-axis* where the function increases and decreases.**

Graph the function g(x) = -|x + 1| and look at the intervals of increase and decrease.

When x = -3 -2 -1 0 1 2

then y = -2 -1 0 -1 -2 -3

The more of the function that you graph, the more you will see that g(x) increases in the interval $(-\infty ,-1)$, where $-\infty $ < x < -1 . g(x) decreases in the interval $(-1,\infty )$, where

-1 < x < $\infty $.

**SAMPLE QUESTIONS**

[13] Identify intervals of increase and decrease for y = x2 – 4.



[14] Identify intervals of increase and decrease for h(x), which is graphed at the right.

 **Lesson 1.5 Intercepts of Functions**

Intercepts of functions are significant in many real-life situations. For example, look at the function a(t) = -90,000 + 600t. This function represents the balance in Saria’s account with Deutche Bank. Saria took out a loan to start her own bistro. She owes the bank $90,000 and has agreed to pay the bank $600 a month. For this function, t = time in months after she borrows the money and a(t) = the balance of her account (the amount she owes).



The **y-intercept** (0, -90,000) represents where Saria’s loan starts, at -$90,000. The **x-intercept** (150, 0) represents where her loan ends: she will have paid off the entire loan in 150 months.

**Notice that the x-intercept is where y = 0 and the**

**y-intercept is where x = 0. You should know to**

**substitute 0 for x to find the y-intercept and vice**

**versa.**

Here is another example: let h(t) = -16t2 + 80t. Here the function represents the height of a toy rocket that is launched from the ground with a starting blast of 80 feet per second into the air; while the blast sends the rocket up, gravity pulls the rocket down at an acceleration of 16 feet per second squared. t = time in seconds and h(t) = height in feet.

The t-intercepts of this function show you when the rocket starts rising and stops falling. The rocket begins at t = 0; the rocket falls back to the ground at t = 5 seconds.

🡪Notice the difference between the abstract function

y = -16x2 + 80x and the actual function h(t) = -16t2 + 80t that fits the real-life rocket. The abstract function for y has the domain of all real numbers, while the real-life function for the rocket has the domain [0, 5], because the rocket starts rising at t = 0 and stops at t = 5.

**SAMPLE QUESTION**

 [15] Find the x-intercept and the y-intercept for the function 20x + 10y = 60, in which x represents the number of adults who attend a concert and y represents the number of children under 12 who attend the concert. Explain what each intercept represents; also explain your reasoning.

 **Lesson 1.6 End Behavior of a Function**

To determine the **end behavior** of a function, you need to notice overall trends—you need to note what y does as x approaches specific values. While every function has its own end behaviors, certain types of functions have behaviors in common.

EXAMPLE 1.6 A

To the right is the graph of y = $\sqrt{9x}$ – 3. You can see one endpoint of the function: (0, -3). However, there is no other endpoint: y continues to increase as x increases. The higher the x value that you plug in, the higher the y value you get out. For this reason, we say that as x🡪$\infty $, y🡪$\infty $. Eventually you will be expected to use a limit to express this end behavior:

The limit, as x approaches infinity, is that y approaches infinity.

$\lim\_{x\to \infty }\sqrt{9x}-3$ = $\infty $.

EXAMPLE 1.6 B

To the right is the graph of the function f(x) = $\frac{1}{x}$.

Here f(x) is a rational function, a fractional one. **IT IS VERY IMPORTANT THAT YOU GET THIS BASIC CONCEPT: DIVIDING BY ZERO IS MEANINGLESS GOBBLEDYGOOK.** Because you cannot divide by zero, x cannot equal zero.



At the right is the graph of f(x). There are four end behaviors we need to name.

[a] As x 🡪 $-\infty $, y 🡪$0$: this means that as x approaches $-\infty $, y approaches $0$. Notice that as x goes farther left, y gets closer to zero high.

[b] As x 🡪 $0^{-}$, y 🡪$-\infty $: this means that as x approaches $0$ from the left, y approaches $-\infty $. Notice that as x goes from $-\infty $ to 0, y goes farther DOWN without bound.

[c] As x 🡪 $0^{+}$, y 🡪$\infty $: this means that as x approaches $0$ from the right, y approaches $\infty $. Notice that as x goes from $\infty $ to 0, y goes farther UP without bound.

[d] As x 🡪 $\infty $, y 🡪$0$: this means that as x approaches $\infty $, y approaches $0$.

**SAMPLE QUESTION**

[16] Name the end behavior of the function graphed at the right.



 **Lesson 1.7 Extrema (Minima and Maxima) of a Function**

A **minimum** is the lowest y value for a function on a given interval for x. A **relative minimum** (the same as a local minimum) is the lowest y value for a specific section of a function’s graph—as long as it is not an endpoint. An **absolute minimum** (the same as a global minimum) is the lowest value of all for a function. Minimum is singular (for one); minima is plural (for more than one).

A **maximum** is the highest value for a function, for a particular section of the graph or the entire graph. A maximum may be relative or absolute. Maximum is singular, while maxima is plural.

High and low function values together are known as **extrema**. Extremum is singular; extrema is plural.



To the right is the graph of f(x). Note: the domain of f(x) is [-5, 4].

f(x) has a lowest point of (0, -2); therefore f(x) has an absolute minimum of -2 (at x = 0).

f(x) has an endpoint at (-5, -1). This is not a relative minimum because it is not the lowest point for a section of the graph beyond itself to the left and right.

f(x) has an absolute maximum of 2 (where x = 4). While (4, 2) is an endpoint, it is also the highest point of all on f(x), so it contains the global maximum of 2.

f(x) has a local maximum of 1 (where x = -3). 1 is the highest y value on the interval (-5, 0) for x.

**SAMPLE QUESTIONS**

Use the graph of g(x) at the right to answer the following questions.

[17] Name the absolute maximum of g(x).

[18] If there are any local maxima of g(x), name them.

[19] Name the absolute minimum of g(x).

[20] If there are any local minima of g(x), name them.

[21] Which points are endpoints and not extrema? Explain your response.

 **Lesson 1.8 Finding Function Values—PLUS Putting it All Together**

Finding a value for f(x) means plugging a value for x into the function and evaluating the expression. Finding an x value for a given function value means plugging a value in for y and solving for x.

EXAMPLE 1.8 A Find f(-8) for f(x) = -x2 – 2.

*Solution*: Plug -8 for x into the formula: f(-8) = -(-8)2 – 2 = -(64) – 2 = -66.

EXAMPLE 1.8 B Given g(x) = $\sqrt{25-x^{2}}$ Find x when g(x) = 3.

*Solution*: Plug 3 in for g(x): 3 = $\sqrt{25-x^{2}}$ . Then solve: Square both sides: 9 = 25 – x2

 Subtract 25: -25 -25

 -16 = -x2

 Multiply by -1: 16 = x2

 Take the square root: 4 or -4 = x.

 \*Check both solutions to see they both work.

**SAMPLE QUESTIONS**

[22] Given h(t) = 2t3 - $\sqrt{3t^{2}-7}$, find h(-2).

[23] Given f(x) = $\sqrt{2x^{2}-2}$, find x when f(x) = 4.

Use the graph of g(x) = -x2 + 4x to answer questions 24-30.

[24] Name the x- and y-intercepts of g(x).

[25] Name any extrema of g(x).

[26] Name the end behavior of g(x).

[27] Name g(-1).

[28] Find x when g(x) = 3.

[29] Name the domain and range of g(x).

[30] Name intervals where g(x) is increasing and where g(x) is decreasing.

[31] Which of the following are functions? [a] x2 – y = 1 [b] x2 – y2 = 1 [c] x – y2 = 1

**SAMPLE QUESTION ANSWERS**

Lesson 1.1

[1] The circle is not a function: it does not pass the vertical line test. Also, when x = 0, y has more than one value, and this disqualifies the graph as a function.

[2] This is not a function: when x = -2, y has more than one value.

[3] This is a function: for each x, y has only one value. (It is okay that y is 2 for each x.)

[4] This is not a function: the graph does not pass the vertical line test.

Lesson 1.2

[5] If you graph the function, you will see that x can go as negative or a positive as you allow, so the domain is this: ($-\infty ,\infty $), the set of all real numbers. The lowest point is (0, -6), so the range is this: ($-6,\infty $).

[6] The graph continues indefinitely left and right, so the domain is ($-\infty ,\infty $). y has only one value, -1, so (-1) is the range.

[7] The radicand 2x + 1 must be zero or greater; if you solve 2x + 1 > 0, x > -0.5; therefore, the domain is ($-0.5,\infty $). The range is ($0,\infty $).

Lesson 1.3

[8] g(-1) = 4. y is 4 when x = -1. [9] g(5) = 4. [10] x = 4 and 0 when g(x) = 3.

[11] y goes down 2 as x goes from 0 to 2: $∆y$ = -2 over this interval.

[12] If you follow the graph going left, you can see it rises as x becomes more negative. Therefore g(-10) > g(-5).

Lesson 1.4

[13] Once you graph the function, you should see it decreases in the interval $(-\infty , 0)$ and increases in the interval $(0, \infty )$.

[14] g(x) increases in the intervals $(-\infty , -3)$ and $(-1, \infty )$. g(x) decreases in the interval (-3, -1).

 Notice you do not need to identify y values: you name values on the x-axis.

Lesson 1.5

[15] If you substitute 0 for x, then y = 6: the y-intercept is (0, 6) and this represents how many children can attend the concert if no adults attend. If you substitute 0 for y, then x = 3: the x-intercept is (3, 0), and this represents how many adults can attend the concert if no children attend.

Lesson 1.6

[16] As x approaches $-\infty $, y approaches $-\infty $; as x approaches $\infty $, y approaches $\infty $.

Lesson 1.7

[17] The absolute maximum is 1 (where x = 0).

[18] There are no relative maxima.

[19] The global minimum is -3 (where x = -4).

[20] There is a local minimum of -1, on the interval 0 < x < 3.

[21] (-6, -1) and (3, 0) are endpoints. Neither contains an absolute maximum or minimum; g(x) does not continue left beyond (-6, -1) or right beyond (3, 0).

Lesson 1.8

[22] h(-2) = 2(-2)3 - $\sqrt{3(-2)^{2}-7}$ = 2(-8) - $\sqrt{12-7}$ = -16 – $\sqrt{5}$. \*This is about -18.24, but the exact value is -16 – $\sqrt{5}$. If you want to look at something pretty, search the internet.

[23] $\sqrt{2x^{2}-2}$ = 4 ; (STEP 1: Square both sides.) 2x2 – 2 = 16; (STEP 2: Add 2.) 2x2 = 18;

(STEP 3: Divide by 2.) x2 = 9; (STEP 4: Take square root.) x = + 3. (STEP 5: Check solutions.)

[24] The x-intercepts are 0 and 4; the y-intercept is 0.

[25] g(x) has a maximum of 4. [The high point is (2, 4).]

[26] Here is the end behavior for g(x): $\lim\_{x\to -\infty }-x^{2}+4x$ = $-\infty $ and $\lim\_{x\to \infty }-x^{2}+4x$ = $-\infty $.

In other words, as x approaches $-\infty $, g(x) approaches $-\infty $, and as x approaches $\infty $, g(x) approaches $-\infty $. As x goes more negative, or more positive, y continues to go farther down.

[27] g(-1) = 3.

[28] When g(x) = 3, x = -1 or 3. The graph passes through (-1, 3) and (3, 3).

[29] The function goes left and right forever, so the domain is $(-\infty ,\infty )$. The graph has a maximum of 4 high, so the range is $(-\infty , 4]$.

[30] g(x) increases in the interval $(-\infty ,2)$; g(x) decreases in the interval $(2,\infty )$. Notice that as a roller coaster, the graph rises until you reach x = 2.

[31] Choice A is a function, because for each x, y can only be one value. Because y is squared in choices B and C, when x is, say, 3, y can either be positive or negative: y can be more than one value. B and C are not functions.